**Part I. Mathematical physics problems**

Well-known classical and generalized forms of solution of mathematical physics problems are considered. The classic solution is a smooth enough function that satisfies the state equations with boundary conditions at each point of its domain. The generalized solution belongs to a Sobolev space and satisfies an integral equality. We consider relations between these notions, its determination from physical laws and methods of its practical finding. However, our general problem is the validity of the method of its definition.

# Chapter 1. Classical models

We consider the classic method of obtaining mathematical models of physical phenomenon. It consists of the choosing of an elementary volume, determining balance relations by means of physical laws, and passage to the limit as this volume shrinks to the point. This relates to the classic solution of mathematical physics problems. This is a smooth enough function that satisfies the state equations and the corresponding boundary conditions at each point of its domain. This notion uses also for finding the approximate solution of the problem. However, we have some doubts about the correctness of the mathematical model definition by the classical method.

## 1.1. Mathematical analysis of a physical phenomenon

Consider the application of mathematical methods for the analysis of a physical phenomenon. In reality, there exists three general stages of this analysis (see Figure 1.1).

1. The definition of the mathematical model;
2. The analysis of the properties of the given model;
3. Practical solving of this model.

We could add here the interpretation of the results of modeling and its practical application. However, these questions already go far beyond mathematics, and this does not apply to the discussed problem. Therefore, we confine ourselves to the three above-mentioned stages.

The definition of the model is based directly from a natural experiment and physical laws. The analysis of the physical phenomenon is a work of the physicist. However, the obtained model is a real mathematical problem. Then theorist mathematician analyses the concrete properties of this model. Particularly, he proves the existence of its solution, the uniqueness of this solution, its functional properties, etc. However, we want not only to determine the general properties of the solution of the problem, but also to find this solution, if it is accurate, then at least approximately. The practical solving of the model is the business of the applied mathematician. Unfortunately, such clear and seemingly quite natural delimitation of actions in the process of researching the problems of mathematical physics sometimes leads to undesirable consequences. The greatest troubles here should be expected at the intersection of individual stages, where the spheres of influence of different specialists are not so clearly determined.



Figure 1.1. The scheme of the research.

At first, we would like to obtain the mathematical model of a physical phenomenon. There exists often the following standard steps of mathematical modeling (see Figure 1.2).

1. The definition of an ***elementary volume*** from the considered set for the analyzing system.
2. The determination of most important ***balance relations*** in this volume by means of physic laws that can be the laws of conservation of the main changing characteristics: energy, momentum, mass, charge, etc.
3. ***Passage to the limit*** at these balance relations as this volume shrinks to a point.

The ***state equations*** are the result of the final step. There are equations with respect to general characteristics of the phenomenon, which are called the ***state functions***. The state equations with initial and boundary conditions give us as a rule the ***classical mathematical models***. Try to realize this idea for a concrete example.



Figure 1.2. Definition of mathematical physics models.

These actions are so customary. Therefore, researchers often do not pay proper attention to one extremely serious procedure. This is the passage to the limit that is one of the most important and at the same time the least obvious mathematical procedures. The problem of the convergence is fundamental for the mathematical analysis. However, in mathematical physics, the difficulties encountered in justification of passage to the limit in the construction of mathematical models are often not given due attention. Note that the insufficient validity of the procedure of the definition of state equations can call into question all subsequent actions on them, in particular, a qualitative and quantitative analysis of the system. Consider a concrete example.

## 1.2. Definition of a mathematical model

We would like to simplify all technical transformations. Therefore, we restrict ourselves to the consideration of an extremely simple and well-known example. Consider the stationary heat transfer phenomenon. Let us have a one-dimensional non-homogeneous body with a length *L* under a source of the heat. A detailed consideration of the determination of the corresponding mathematical model can be found in any classical course in mathematical physics. We give subsequent transformations because of the strong need to get a clear idea of the essence of our problem.

Choose an interval [*x*,*x+h*] as an elementary volume for the one-dimensional case. Determine the change of the quantity of the heat *q* there. We have the following equality

 **** (1.1)

The known function *f* characterizes the source of the heat here. The function *f* determines the heat on the unit interval under this source. Therefore, the integral of the right-hand side of the last equality is the heat at the given interval of the body under the source (see Figure 1.3). Let the dimension of the heat flux be the same as the dimension of the spatial coordinate *x.* It is equal to the difference between the quantity of the heat at the begin and at the end of the interval because of the equality (1.1).

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Figure 1.3. Change of the heat quantity.

The existence of the heat flux at a point *x* is the corollary of the difference between the temperature at this point and at the previous site. This is the basis of the heat conductivity phenomenon. By the ***Fourier law***, the heat flux at a point *x* is proportional to the difference between the temperature at begin of the given interval and at its end. Then it is inverse proportional to the length of this interval. Besides, the heat moves from the domain with a large temperature to the domain with a small temperature. Therefore, we obtain the formula (see Figure 1.4)

  ****  (1.2)

where *u* is the temperature, and *k* is the coefficient of the heat conductivity of the body. The temperature is our state function, because it describes the state of the considered system; and *k* is a known function. The length of the intervals at the equalities (1.1) and (1.2) can be different. However, we consider easiest case of its equality.

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Figure 1.4. Fourier law.

Suppose the function *u* is twice continuously differentiable at the given interval. After the passage to the limit at the equalities (1.1), (1.2) as *h* → 0, we get the ***non-homogeneous stationary one-dimensional heat equation***

  (1.3)

We suppose also that the temperature of the body on the boundary of the interval is equal to zero here. Then we have the boundary conditions

 *u*(0) = 0*, u*(*L*) = 0. (1.4)

**Definition 1.1**. *The boundary problem* (1.3), (1.4) *is the* ***classical mathematical model*** *of the considered phenomenon.*

Consider the properties of this model.

## 1.3. Classical solution of the system

We have the second order differential equation (1.3) with respect to the function *u* that satisfies also the homogeneous boundary conditions (1.4). Choose the natural functional class for this solution (see Figure 1.5). Determine the set  of twice continuously differentiable function on the interval  that is equal to the zero at the ends of this interval. Note that this is the Banach space. We shell use the shorter denotation . Determine the standard form of the solution for the boundary problem (1.3), (1.4).

**Definition 1.2**. *The function*  *from the set*  *is called the* ***classical solution*** *of the boundary problem* (1.3), (1.4), *if it satisfies the equality* (1.3) *for all point x of the open interval* .



Figure 1.5. Choice of the functional class for the classical solution.

Note that the definition of the mathematical model is associated to the notion of the classical solution of the considered mathematical physics problem because the possibility of passage to the limit at the balance relations requires the functional properties of the classical solution for the considered state function.

We have also an interest to the qualitative and quantitative analysis of this mathematical model. The qualitative analysis is, at first, the proof of solvability of the system. It is necessary to prove that the boundary problem (1.3), (1.4) has a classical solution under some properties of the known functions *f* and *k*. We could determine also the uniqueness of the solution, its continuous dependence from the parameters of the systems, etc. These results can be obtained with using the differential equations theory, in reality. However, we do not determine these properties now.

Consider now the final step of the analysis. This is practical solving of the boundary problem.

## 1.4. Approximate solution of the system

The best result of the analysis of the considered problem could be obtaining the analytic formula of its solution. This is the direct dependence of the temperature *u* from the space coordinate *x* for all value of all parameters of the system, namely the length of the body *L*, the heat source *f*, and the heat conductivity *k.* Unfortunately, the analytic solution of the mathematical physics problems can be obtained for easy enough partial cases only. However, we can find the approximate solution of the problems for the general case. There exist many approximate methods of solving for the problem (1.3), (1.4). We choose the ***finite difference method*** because of its high enough effectiveness and simplicity. It uses the division of the given domain of the state function to parts and the approximation of its derivatives by the corresponding differences. Determine at first the easiest formulas of the approximate differentiation.

Consider a function **** Suppose it is differentiable. Then we have the equality

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by the Taylor formula, where  as . If the value *h* is small enough, we have the approximate equality

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Therefore, we can find

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This formula of the approximate differentiation is called the ***forward difference***. Its exactness is determined by the value of the step *h.*

Analogically, from the formula with negative step

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it follows the approximate equality

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Then we get

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This formula of the approximate differentiation is called the ***back difference***.

Using the formulas of approximate differentiation, determine the algorithm of finding an approximate solution of the boundary problem. Divide the given interval (0,*L*) into *M* equal parts. Choose the step *h = L*/*M* and the points **** Determine the standard ***difference operators*** on the Euclid spaces

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by the equalities

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where ****

Consider the equality (1.1) at the arbitrary point *xi*. We have

 **** (1.5)

where **** Analogically,from the equality (1.2) it follows that

 **** (1.6)

where  Put the value *qi* from the equality (1.6) to (1.5). We obtain

 **** (1.7)

where



We use the difference equation (1.7) as the approximation of the differential equation (1.3). Add also the boundary conditions

 *u*0 = 0, *uM* = 0. (1.8)

The system of the linear algebraic equations (1.7), (1.8) can be solved by means of standard methods, for example, by ***marching method***. Therefore, we find all values *ui*, namely the ***grid function***.

Note that our solution is the function of the continuous argument. Then we use the linear interpolation of the grid function (see Figure 1.6)

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It satisfies the equality



Besides, the function *uh* is equal to zero on the boundary of our domain.

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Figure 1.6. Linear interpolation of the grid function.

We would like use *uh* for a small enough value *h* as an approximate solution of the considered problem. Therefore, it is necessary to prove the convergence *uh* → *u* as *h* → 0. This is the convergence in the class of the twice continuously differentiable functions, because we would like to determine the classical solution of our boundary problem that is an element of this functional space. This is the substantiation of the numerical method and the basis of the practical solution of the problem.

**Remark 1.1**. We shall return to the convergence of the finite difference method in Caption 9.

Note that the properties of the classical solution are used for all three steps of the research (see Figure 1.7). Indeed, the functional class  is on the base of the notion of the classical solution. We use it also for passage to the limit at the balance relations. Besides, it was be used for the determination and the substantiation of the method of finding the approximate solution of the problem. This indicates the existence of a serious connection between the different stages of solving of mathematical physics problems.



Figure 1.7. Assumption about the state function for the classical method.

**Remark 1.2**. We have to make sure that this connection is extremely deep. Moreover, the separation of the research process into stages is conditional.

Now we consider the central problem of this course. This is the validity of the process of obtaining this mathematical model.

## 1.5. Validity of the classic method

We obtained our mathematical model by means of the passage to the limit at the balance relations as the length of the elementary interval tends to zero. As is known, the limit is the general notion of mathematical analysis. The convergence is one of the most important and difficult mathematical operations, in the realization of which it is necessary to have maximum caution. This operation is surprisingly effective, but remains mysterious, because of its direct connection with infinite procedures. Therefore, one certainly requires a rigorous justification. The justification of the convergence is often the main technical difficulty at the theory of partial differential equations, computational mathematics, the theory of optimal control, and other branches of mathematics.

We would like pass to the limit at the equalities (1.1), (1.2). However, why this limit exists? Our analysis is correct, if the state function *u* is twice continuously differentiable. Indeed, we used before this hypothesis. Therefore, we have next question. Why this function satisfies this property?

Maybe, we could use physical reasons. Perhaps, the temperature of the body must be a smooth function. However, our body can be non-homogeneous by our suppositions. Moreover, the heat source can have different properties. We do not know if the temperature is twice continuously differentiable for these cases. Besides, our object of analysis is the mathematical model, but not a physical body. We do not know if the properties of the phenomenon and its model are same. Therefore, we need to use mathematical reasons only.

Maybe, we could prove the desired property of the considered function directly. Indeed, we could use the results of the differential equations theory. In reality, the solution of the given boundary problem is twice continuously differentiable under some assumptions with respect to the known functions *k* and *f*. This is the result of the second stage of the analysis. This is obtained by the proof of the existence of the classical solution of the boundary problem. Unfortunately, this result can be obtained after the determination of the mathematical model only. We cannot any possibility to analyze the equation before its definition. This is the strange enough result. The determination of the model uses the properties of its state function. However, it is necessary to have the model for the analysis of this state function (see Figure 1.8).



Figure 1.8. Relations between obtaining of the mathematical model and properties of state functions.

This considered difficulty often remains without due attention. This is explained, apparently, by the fact that physicists, as well as chemists, biologists, economists, etc. are engaged in generally in immediate modeling of nature processes. Indeed, the main problem for them is to identify those reasons that predetermined the considered events. They establish the qualitative and quantitative impact of each of the considered factors on the result. In the our case, it is important for a physicist that the investigated body is sufficiently long and thin, the characteristics of the phenomenon do not change, heat transfer is due solely to heat conduction, besides, convection, radiation, heat exchange with the environment, the presence of chemical reactions and other possible causes of heat transfer, do not important here. The physicist tries to understand the studied phenomenon and often does not think about the technical mathematical difficulties. Indeed, the justification of mathematical procedures is the business of the professional mathematicians.

On the other hand, professional mathematicians usually consider the equations as an immediate given object of research. They, as a rule, believe that the derivation of equations is the work of physicists. Indeed, we have the statement of the mathematical problem, for example, a boundary problem for a differential equation. Mathematicians need to prove the existence of the given problem and find the algorithm of its solving. An equation is a really existing object for theoretical mathematicians, as natural as an electrical circuit for physicists, a chemical reagent for chemists, and a living organism for a biologists. The equations are the serious objects of analysis for mathematicians if it do not have any physical sense even.

We could suppose that this problem is not very important. Indeed, the absolute majority of physicists and mathematicians is not very interested, apparently, in its solution. Maybe this is really so. However, if we have the right to carry out formally without a rigorous mathematical justification one of the stages of the analysis of the problem, then we must also apply to its other stages. Is it necessary to prove the existence of a solution to a problem at all, if already from physical considerations it follows, for example, that the body at any point possesses a certain temperature? Therefore, the required solution must certainly exist, and there is no problem here. Many researchers of practice reason this way... Indeed, we can write down and program all the necessary formulas, and the computer will definitely produce some result... However, how do we know that this is indeed the solution of the given equations? Of course, we can compare the results with experiment. However, we do not sure that our model describes the considered phenomenon exactly. Besides, we do not know exactly the parameters of the system, for example, the functions *k* and *f* in our example. Therefore, it is not clear how we can use the results of experiment for proving the existence of the solution. We can use results of computing. However, can we guaranty that we found the approximate solution of the problem without the proof of the convergence of the numerical algorithm? If the mathematical rigor is needed at the stage of qualitative and quantitative analysis of the system, then, apparently, it is needed too in the construction of a mathematical model. It is better to find an approximate solution of the problem without proving its solvability and convergence of the algorithm, than not to obtain any result. However, this is the evidence of our weakness only.

We apparently have good reasons for doubting the correctness of the derivation of mathematical models by available means. This exceptionally serious circumstance can cast doubt on the validity of the classical approach in mathematical physics. Therefore, there is a need to find another way of constructing mathematical models that do not possess these disadvantages. In particular, one can try to pass from the concept of the classical solution of the problem to a generalized solution. As we know, the most important directions in the development of mathematical physics are connected for a long time already with the generalized, and not with the classical approach. Therefore, we suppose that this could be the basis of the correct constructing of mathematical models.

## Conclusions

* Mathematical analysis of physical phenomenon contains the definition of the mathematical model, the determination of the properties of state functions and finding of the solution of the problem.
* All stages of this analysis are based on the classic solution of the problem.
* The classic solution of the problem is a twice differentiable function that satisfies the state equation at the arbitrary point and the boundary conditions.
* The classic mathematical model is the result of passage to the limit at the balance relations that are the corollary of physical laws.
* The limit exists if the state function is twice differentiable.
* The necessary properties of the state function can be obtained by means of the differential equations theory after the determination of the mathematical model.
* We cannot any information about the state function before the determination of the mathematical model.
* The classic method of the determination of the mathematical model is not substantiated.
* It is necessary to find another method of analysis.

The modern theory of equations of mathematical physics often uses the concept of a generalized solution of the problem in place of the classical one. Therefore, we shell try to substantiate the process of mathematical modeling by means of a generalized approach.